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Real hypersurfaces in complex hyperbolic two-plane Grassmannians with Reeb invariant Ricci tensor ☆

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In this paper we first introduce the full expression of the curvature tensor of a real hypersurface M in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 2$ from the equation of Gauss. Next we derive a new formula for the Ricci tensor S of M in $SU_{2,m}/S(U_2 \cdot U_m)$. Finally we give a complete classification of Hopf hypersurfaces in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2 \cdot U_m)$ with Reeb invariant Ricci tensor, that is, $\mathcal{L}_\xi S = 0$. Each can be described as a tube over a totally geodesic $SU_{2,m-1}/S(U_2 \cdot U_{m-1})$ in $SU_{2,m}/S(U_2 \cdot U_m)$ or a horosphere whose center at infinity is singular.

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1. Introduction

In the geometry of real hypersurfaces in complex space forms $M_m(c)$ or in quaternionic space forms $Q_m(c)$ Kimura [7,8] (resp. Pérez and Suh [10]) considered real hypersurfaces in $M_n(c)$ (resp. in $Q_m(c)$) with commuting Ricci tensor, that is, $S\phi = \phi S$, (resp. $S\phi_i = \phi_i S$, $i = 1, 2, 3$) where S and ϕ (resp. S and ϕ_i, $i = 1, 2, 3$) denote the Ricci tensor and the structure tensor of real hypersurfaces in $M_m(c)$ (resp. in $Q_m(c)$).

In [7,8], Kimura has classified that a Hopf hypersurface M in complex projective space $P_m(\mathbb{C})$ with commuting Ricci tensor is locally congruent to of type (A), a tube over a totally geodesic $P_k(\mathbb{C})$, of type (B), a tube over a complex quadric Q_{m-1}, $\cot^2 2r = m - 2$, of type (C), a tube over $P_1(\mathbb{C}) \times P_{(m-1)/2}(\mathbb{C})$, $\cot^2 2r = \frac{1}{m-2}$ and n is odd, of type (D), a tube over a complex two-plane Grassmannian $G_2(\mathbb{C}^5)$, $\cot^2 2r = \frac{3}{5}$ and $n = 9$, of type (E), a tube over a Hermitian symmetric space $SO(10)/U(5)$, $\cot^2 2r = \frac{5}{9}$ and $m = 15$.

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On the other hand, in a quaternionic projective space $\mathbb{Q}P^m$ Pérez and Suh [10] have classified real hypersurfaces in QP^m with commuting Ricci tensor $S\phi_i = \phi_i S$, $i = 1, 2, 3$, where S (resp. ϕ_i) denotes the Ricci tensor (resp. the structure tensor) of M in $\mathbb{Q}P^m$, is locally congruent to of A_1, A_2-type, that is, a tube over $\mathbb{Q}P^k$ with radius $0 < r < \frac{\pi}{2}$, $k \in \{0, \cdots, m-1\}$. The almost contact structure vector fields $\{\xi_1, \xi_2, \xi_3\}$ are defined by $\xi_i = -J_i N$, $i = 1, 2, 3$, where J_i, $i = 1, 2, 3$, denote a quaternionic Kähler structure of $\mathbb{Q}P^m$ and N a unit normal field of M in $\mathbb{Q}P^m$. Moreover, Pérez and Suh [9] have considered the notion of $\nabla_{\xi_i} R = 0$, $i = 1, 2, 3$, where R denotes the curvature tensor of a real hypersurface M in $\mathbb{Q}P^m$, and proved that M is locally congruent to a tube of radius $\frac{\pi}{4}$ over $\mathbb{Q}P^k$.

Let us denote by $SU_{2,m}$ the set of $(m+2)\times(m+2)$-indefinite special unitary matrices and U_m the set of $m\times m$-unitary matrices. Then the Riemannian symmetric space $SU_{2,m}/S(U_2U_m)$, $m \geq 2$, which consists of complex two-dimensional subspaces in indefinite complex Euclidean space $\mathbb{C}_2^{m+2}$, has a remarkable feature that it is a Hermitian symmetric space as well as a quaternionic Kähler symmetric space. In fact, among all Riemannian symmetric spaces of noncompact type the symmetric spaces $SU_{2,m}/S(U_2U_m)$, $m \geq 2$, are the only ones which are Hermitian symmetric and quaternionic Kähler symmetric.

The existence of these two structures leads to a number of interesting geometric problems on $SU_{2,m}/S(U_2U_m)$, one of which we are going to study in this article. To describe this problem, we denote by J the Kähler structure and by $\mathfrak{J}$ the quaternionic Kähler structure a quaternionic Kähler structure $\mathfrak{J}$ not containing J on $SU_{2,m}/S(U_2U_m)$ defined by $\mathfrak{J} = \text{Span}\{J_1, J_2, J_3\}$. Let M be a connected hypersurface in $SU_{2,m}/S(U_2U_m)$ and denote by N a unit normal to M. Then a structure vector field ξ defined by $\xi = -JN$ is said to be a Reeb vector field.

Next, we consider the standard embedding of $SU_{2,m-1}$ in $SU_{2,m}$. Then the orbit $SU_{2,m-1} \cdot o$ of $SU_{2,m-1}$ through o is the Riemannian symmetric space $SU_{2,m-1}/S(U_2U_{m-1})$ embedded in $SU_{2,m}/S(U_2U_m)$ as a totally geodesic submanifold. Every tube around $SU_{2,m-1}/S(U_2U_{m-1})$ in $SU_{2,m}/S(U_2U_m)$ has the property that both maximal complex subbundle $\mathcal{C}$ and quaternionic subbundle $\mathcal{Q}$ are invariant under the shape operator.

Finally, let m be even, say $m = 2n$, and consider the standard embedding of $Sp_{1,n}$ in $SU_{2,2n}$. Then the orbit $Sp_{1,n} \cdot o$ of $Sp_{1,n}$ through o is the quaternionic hyperbolic space $\mathbb{H}H^n$ embedded in $SU_{2,2n}/S(U_2U_{2n})$ as a totally geodesic submanifold. Any tube around $\mathbb{H}H^n$ in $SU_{2,2n}/S(U_2U_{2n})$ has the property that both $\mathcal{C}$ and $\mathcal{Q}$ are invariant under the shape operator.

As a converse of the statements mentioned above, we assert that with one possible exceptional case there are no other such real hypersurfaces. Related to such a result and the work in Eberlein [4], we introduce another theorem due to Berndt and Suh [3] as follows:

Theorem A. *Let M be a connected hypersurface in $SU_{2,m}/S(U_2U_m)$, $m \geq 2$. Then the maximal complex subbundle $\mathcal{C}$ of TM and the maximal quaternionic subbundle $\mathcal{Q}$ of TM are both invariant under the shape operator of M if and only if M is congruent to an open part of one of the following hypersurfaces:*

(A) *a tube around a totally geodesic $SU_{2,m-1}/S(U_2U_{m-1})$ in $SU_{2,m}/S(U_2U_m)$;*
(B) *a tube around a totally geodesic $\mathbb{H}H^n$ in $SU_{2,2n}/S(U_2U_{2n})$, $m = 2n$;*
(C) *a horosphere in $SU_{2,m}/S(U_2U_m)$ whose center at infinity is singular;*

or the following exceptional case holds:

(D) *The normal bundle νM of M consists of singular tangent vectors of type $JX \perp \mathfrak{J}X$. Moreover, M has at least four distinct principal curvatures, three of which are given by*

$$\alpha = \sqrt{2}\ ,\ \gamma = 0\ ,\ \lambda = \frac{1}{\sqrt{2}}$$

with corresponding principal curvature spaces

$$T_\alpha = TM \ominus (\mathcal{C} \cap \mathcal{Q}) \ , \ T_\gamma = J(TM \ominus \mathcal{Q}) \ , \ T_\lambda \subset \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}.$$

If μ is another (possibly nonconstant) principal curvature function, then we have $T_\mu \subset \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}$, $JT_\mu \subset T_\lambda$ and $\mathfrak{J}T_\mu \subset T_\lambda$.

In Theorem A the maximal complex subbundle $\mathcal{C}$ of TM is invariant under the shape operator if and only if the Reeb vector field ξ becomes a principal vector field for the shape operator A of M in $SU_{2,m}/S(U_2 U_m)$. In this case the Reeb vector field ξ is said to be a Hopf vector field. The flow generated by the integral curves of the structure vector field ξ for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ is said to be a *geodesic Reeb flow*.

In the proof of Theorem A we proved that the 1-dimensional distribution $[\xi]$ is contained in either the 3-dimensional distribution $\mathcal{Q}^\perp$ or in the orthogonal complement $\mathcal{Q}$ such that $T_xM = \mathcal{Q} \oplus \mathcal{Q}^\perp$. The case (A) in Theorem A is just the case that the 1-dimensional distribution $[\xi]$ belongs to the distribution $\mathcal{Q}$. Of course, it is not difficult to check that the Ricci tensor S of type (A) or of type (C) with $JX \in \mathfrak{J}X$ in Theorem A commutes with the structure tensor, that is $S\phi = \phi S$. Then it must be a natural question to ask whether real hypersurfaces in $SU_{2,m}/S(U_2 \cdot U_m)$ with commuting Ricci tensor can exist or not.

On the other hand, in due to [19] Suh has considered such a converse problem and has given a complete classification of real hypersurfaces in $SU_{2,m}/S(U_2 \cdot U_m)$ satisfying $S\phi = \phi S$ as follows:

Theorem B. *Let M be a Hopf hypersurface in $SU_{2,m}/S(U_2 \cdot U_m)$ with commuting Ricci tensor, $m \geq 3$. Then M is locally congruent to an open part of a tube around some totally geodesic $SU_{2,m-1}/S(U_2 \cdot U_{m-1})$ in $SU_{2,m}/S(U_2 \cdot U_m)$ or a horosphere whose center at infinity with $JX \in \mathfrak{J}X$ is singular.*

In a compact complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ we have considered the notion of Ricci commuting [12], $S\phi = \phi S$, and give a characterization of type (A), which is a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ and have proved a nonexistence property for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with *parallel Ricci tensor* in [14]. Then, naturally, we can consider more general notions like *Reeb invariant*, semi-parallel, harmonic curvature, and *Reeb parallel* which are given by $\mathcal{L}_\xi S = 0$, $R(X,Y)S = 0$, $(\nabla_X S)Y = (\nabla_Y S)X$, $\nabla_\xi S = 0$, for any vector fields X and Y and the Reeb vector field ξ on real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ respectively, where $R(X,Y)$ and S denote the curvature tensor and the Ricci tensor of M in $G_2(\mathbb{C}^{m+2})$ (see [13,15,16]). These conditions are weaker than usual notion of *parallel Ricci* tensor.

Motivated by such notions for M in $G_2(\mathbb{C}^{m+2})$, recently, Suh and Woo [20] have considered the notion of *Ricci parallel*, that is, $\nabla S = 0$ in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$, and proved that there do not exist any hypersurfaces in $SU_{2,m}/S(U_2 \cdot U_m)$ with *parallel Ricci* tensor.

As mentioned in Theorem B, when a real hypersurface M in $SU_{2,m}/S(U_2 \cdot U_m)$ is locally congruent to an open part of a tube over a totally geodesic $SU_{2,m-1}/S(U_2 \cdot U_{m-1})$ in $SU_{2,m}/S(U_2 \cdot U_m)$ or a horosphere, the Reeb vector field becomes $\xi = \xi_1$, and it becomes $\mathcal{L}_\xi A = 0$ (see [21]). From this, together with the other formula $\mathcal{L}_\xi \phi = 0$, $\mathcal{L}_\xi \phi_1 = 0$, and

$$(\mathcal{L}_\xi \eta_2) \otimes \xi_2 + \eta_2 \otimes \mathcal{L}_\xi \xi_2 + (\mathcal{L}_\xi \eta_3) \otimes \xi_3 + \eta_3 \otimes \mathcal{L}_\xi \xi_3 = 0.$$

Then it can be easily checked that they satisfy $\mathcal{L}_\xi S = 0$. In this case, we say that M has a *Reeb invariant Ricci tensor*.

From such a point of view, conversely, let us consider a Hopf hypersurface in $SU_{2,m}/S(U_2 \cdot U_m)$ with Reeb invariant Ricci tensor, that is, $\mathcal{L}_\xi S = 0$. Then naturally, the purpose of this paper is to show *Ricci commuting* if the Ricci tensor is *Reeb invariant*. Then by virtue of Theorem B we assert the following

Main Theorem. *Let M be a Hopf hypersurface in $SU_{2,m}/S(U_2{\cdot}U_m)$ with Reeb invariant Ricci tensor, $m{\geq}3$. Then M is locally congruent to an open part of a tube around some totally geodesic $SU_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ or a horosphere whose center at infinity with $JX{\in}\mathfrak{J}X$ is singular.*

A remarkable consequence of our Main Theorem is that a connected complete real hypersurface in $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 3$ with Reeb invariant Ricci tensor is homogeneous and has a commuting Ricci tensor. This was also true in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, which could be identified with symmetric space of compact type $SU_{m+2}/S(U_2{\cdot}U_m)$, as follows from the classification. It would be interesting to understand the actual reason for it (see [1,2,9,11,12]).

2. The complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2{\cdot}U_m)$

In this section we summarize basic material about complex hyperbolic Grassmann manifolds $SU_{2,m}/S(U_2{\cdot}U_m)$, for details we refer to [1–3,5,6,12,17,18].

The Riemannian symmetric space $SU_{2,m}/S(U_2{\cdot}U_m)$, which consists of all complex two-dimensional linear subspaces in indefinite complex Euclidean space $\mathbb{C}_2^{m+2}$, becomes a connected, simply connected, irreducible Riemannian symmetric space of noncompact type and with rank two. Let $G = SU_{2,m}$ and $K = S(U_2{\cdot}U_m)$, and denote by $\mathfrak{g}$ and $\mathfrak{k}$ the corresponding Lie algebra of the Lie group G and K respectively. Let B be the Killing form of $\mathfrak{g}$ and denote by $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to B. The resulting decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$. The Cartan involution $\theta \in \mathrm{A}ut(\mathfrak{g})$ on $\mathfrak{su}_{2,m}$ is given by $\theta(A) = I_{2,m}AI_{2,m}$, where

$$I_{2,m} = \begin{pmatrix} -I_2 & 0_{2,m} \\ 0_{m,2} & I_m \end{pmatrix}$$

I_2 and I_m denotes the identity $(2{\times}2)$-matrix and $(m{\times}m)$-matrix respectively. Then $< X, Y >= -B(X, \theta Y)$ becomes a positive definite $\mathrm{A}d(K)$-invariant inner product on $\mathfrak{g}$. Its restriction to $\mathfrak{p}$ induces a metric g on $SU_{2,m}/S(U_2{\cdot}U_m)$, which is also known as the Killing metric on $SU_{2,m}/S(U_2{\cdot}U_m)$. Throughout this paper we consider $SU_{2,m}/S(U_2{\cdot}U_m)$ together with this particular Riemannian metric g.

The Lie algebra $\mathfrak{k}$ decomposes orthogonally into $\mathfrak{k} = \mathfrak{su}_2 \oplus \mathfrak{su}_m \oplus \mathfrak{u}_1$, where $\mathfrak{u}_1$ is the one-dimensional center of $\mathfrak{k}$. The adjoint action of $\mathfrak{su}_2$ on $\mathfrak{p}$ induces the quaternionic Kähler structure $\mathfrak{J}$ on $SU_{2,m}/S(U_2{\cdot}U_m)$, and the adjoint action of

$$Z = \begin{pmatrix} \frac{mi}{m+2} I_2 & 0_{2,m} \\ 0_{m,2} & \frac{-2i}{m+2} I_m \end{pmatrix} \in \mathfrak{u}_1$$

induces the Kähler structure J on $SU_{2,m}/S(U_2{\cdot}U_m)$.

We identify the tangent space $T_oSU_{2,m}/S(U_2{\cdot}U_m)$ of $SU_{2,m}/S(U_2{\cdot}U_m)$ at o with $\mathfrak{p}$ in the usual way. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Since $SU_{2,m}/S(U_2{\cdot}U_m)$ has rank two, the dimension of any such subspace is two. Every nonzero tangent vector $X \in T_oSU_{2,m}/S(U_2{\cdot}U_m) \cong \mathfrak{p}$ is contained in some maximal abelian subspace of $\mathfrak{p}$. Generically this subspace is uniquely determined by X, in which case X is called regular.

If there exists more than one maximal abelian subspaces of $\mathfrak{p}$ containing X, then X is called singular. There is a simple and useful characterization of the singular tangent vectors: A nonzero tangent vector $X \in \mathfrak{p}$ is singular if and only if $JX \in \mathfrak{J}X$ or $JX \perp \mathfrak{J}X$.

Up to scaling there exists a unique $S(U_2{\cdot}U_m)$-invariant Riemannian metric g on $SU_{2,m}/S(U_2{\cdot}U_m)$. Equipped with this metric $SU_{2,m}/S(U_2{\cdot}U_m)$ is a Riemannian symmetric space of rank two which is both Kähler and quaternionic Kähler.

For computational reasons we normalize g such that the minimal sectional curvature of $(SU_{2,m}/S(U_2{\cdot}U_m), g)$ is -4. The sectional curvature K of the noncompact symmetric space $SU_{2,m}/S(U_2{\cdot}U_m)$ equipped with the Killing metric g is bounded by $-4\le K\le 0$. The sectional curvature -4 is obtained for all 2-planes $\mathbb{C}X$ when X is a nonzero vector with $JX\in\mathfrak{J}X$.

When $m = 1$, $G_2^*(\mathbb{C}^3) = SU_{1,2}/S(U_1{\cdot}U_2)$ is isometric to the two-dimensional complex hyperbolic space $\mathbb{C}H^2$ with constant holomorphic sectional curvature -4.

When $m = 2$, we note that the isomorphism $SO(4,2) \simeq SU(2,2)$ yields an isometry between $G_2^*(\mathbb{C}^4) = SU_{2,2}/S(U_2{\cdot}U_2)$ and the indefinite real Grassmann manifold $G_2^*(\mathbb{R}_2^6)$ of oriented two-dimensional linear subspaces of an indefinite Euclidean space $\mathbb{R}_2^6$. For this reason we assume $m \geq 3$ from now on, although many of the subsequent results also hold for $m = 1, 2$.

The Riemannian curvature tensor $\bar{R}$ of $SU_{2,m}/S(U_2{\cdot}U_m)$ is locally given by

$$\begin{aligned}\bar{R}(X,Y)Z = -\frac{1}{2}\Big[&g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ &- g(JX,Z)JY - 2g(JX,Y)JZ \\ &+ \sum_{\nu=1}^{3}\{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y \\ &- 2g(J_\nu X,Y)J_\nu Z\} \\ &+ \sum_{\nu=1}^{3}\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\}\Big],\end{aligned} \tag{2.1}$$

where J_1, J_2, J_3 is any canonical local basis of $\mathfrak{J}$.

3. Real hypersurfaces in $SU_{2,m}/S(U_2{\cdot}U_m)$

Let M be a real hypersurface in $SU_{2,m}/S(U_2{\cdot}U_m)$, that is, a submanifold in $SU_{2,m}/S(U_2{\cdot}U_m)$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Levi Civita covariant derivative of (M, g). We denote by $\mathcal{C}$ and $\mathcal{Q}$ the maximal complex and quaternionic subbundle of the tangent bundle TM of M, respectively. Now let us put

$$JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N \tag{3.1}$$

for any tangent vector field X of a real hypersurface M in $SU_{2,m}/S(U_2{\cdot}U_m)$, where ϕX denotes the tangential component of JX and N a unit normal vector field of M in $SU_{2,m}/S(U_2{\cdot}U_m)$.

From the Kähler structure J of $SU_{2,m}/S(U_2{\cdot}U_m)$ there exists an almost contact metric structure (ϕ, ξ, η, g) induced on M in such a way that

$$\phi^2 X = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ \phi\xi = 0, \quad \text{and} \quad \eta(X) = g(X, \xi) \tag{3.2}$$

for any vector field X on M and $\xi = -JN$.

If M is orientable, then the vector field ξ is globally defined and said to be the induced *Reeb vector field* on M. Furthermore, let J_1, J_2, J_3 be a canonical local basis of $\mathfrak{J}$. Then each J_ν induces a local almost contact metric structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$, $\nu = 1, 2, 3$, on M. Locally, $\mathcal{C}$ is the orthogonal complement in TM of the real span of ξ, and $\mathcal{Q}$ the orthogonal complement in TM of the real span of $\{\xi_1, \xi_2, \xi_3\}$.

Furthermore, let $\{J_1, J_2, J_3\}$ be a canonical local basis of $\mathfrak{J}$. Then the quaternionic Kähler structure J_ν of $SU_{2,m}/S(U_2{\cdot}U_m)$, together with the condition

$$J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$$

in section 1, induces an almost contact metric 3-structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M as follows:

$$\begin{aligned}
&\phi_\nu^2 X = -X + \eta_\nu(X)\xi_\nu,\ \phi_\nu\xi_\nu = 0,\ \eta_\nu(\xi_\nu) = 1\\
&\phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu\xi_{\nu+1} = \xi_{\nu+2},\\
&\phi_\nu\phi_{\nu+1}X = \phi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu,\\
&\phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1}
\end{aligned} \tag{3.3}$$

for any vector field X tangent to M. The tangential and normal component of the commuting identity $JJ_\nu X = J_\nu JX$ give

$$\phi\phi_\nu X - \phi_\nu\phi X = \eta_\nu(X)\xi - \eta(X)\xi_\nu \text{ and } \quad \eta_\nu(\phi X) = \eta(\phi_\nu X). \tag{3.4}$$

The last equation implies $\phi_\nu\xi = \phi\xi_\nu$. The tangential and normal component of $J_\nu J_{\nu+1}X = J_{\nu+2}X = -J_{\nu+1}J_\nu X$ give

$$\phi_\nu\phi_{\nu+1}X - \eta_{\nu+1}(X)\xi_\nu = \phi_{\nu+2}X = -\phi_{\nu+1}\phi_\nu X + \eta_\nu(X)\xi_{\nu+1} \tag{3.5}$$

and

$$\eta_\nu(\phi_{\nu+1}X) = \eta_{\nu+2}(X) = -\eta_{\nu+1}(\phi_\nu X). \tag{3.6}$$

Putting $X = \xi_\nu$ and $X = \xi_{\nu+1}$ into the first of these two equations yields $\phi_{\nu+2}\xi_\nu = \xi_{\nu+1}$ and $\phi_{\nu+2}\xi_{\nu+1} = -\xi_\nu$ respectively. Using the Gauss and Weingarten formulas, the tangential and normal component of the Kähler condition $(\bar\nabla_X J)Y = 0$ give $(\nabla_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi$ and $(\nabla_X\eta)Y = g(\phi AX, Y)$. The last equation implies $\nabla_X\xi = \phi AX$. Finally, using the explicit expression for the Riemannian curvature tensor $\bar R$ of $SU_{2,m}/S(U_2{\cdot}U_m)$ in [3] the Codazzi equation takes the form

$$\begin{aligned}
(\nabla_X A)Y - (\nabla_Y A)X = -\frac{1}{2}\Big[&\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\\
&+ \sum_{\nu=1}^{3}\big\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu\big\}\\
&+ \sum_{\nu=1}^{3}\big\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\big\}\\
&+ \sum_{\nu=1}^{3}\big\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\big\}\xi_\nu\Big]
\end{aligned} \tag{3.7}$$

for any vector fields X and Y on M. Moreover, by the expression of the curvature tensor (2.1), we have the equation of Gauss as follows:

$$\begin{aligned}
R(X, Y)Z = -\frac{1}{2}\Big[&g(Y, Z)X - g(X, Z)Y\\
&+ g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\\
&+ \sum_{\nu=1}^{3}\{g(\phi_\nu Y, Z)\phi_\nu X - g(\phi_\nu X, Z)\phi_\nu Y - 2g(\phi_\nu X, Y)\phi_\nu Z\}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\nu=1}^{3}\{g(\phi_\nu\phi Y,Z)\phi_\nu\phi X-g(\phi_\nu\phi X,Z)\phi_\nu\phi Y\}\\
&-\sum_{\nu=1}^{3}\{\eta(Y)\eta_\nu(Z)\phi_\nu\phi X-\eta(X)\eta_\nu(Z)\phi_\nu\phi Y\}\\
&-\sum_{\nu=1}^{3}\{\eta(X)g(\phi_\nu\phi Y,Z)-\eta(Y)g(\phi_\nu\phi X,Z)\}\,\xi_\nu\Big]\\
&+g(AY,Z)AX-g(AX,Z)AY \qquad (3.8)
\end{aligned}
$$

for any vector fields X, Y, and Z on M. Here after, unless otherwise stated, we want to use these basic equations mentioned above frequently without referring to them explicitly.

4. Some preliminaries in $SU_{2,m}/S(U_2{\cdot}U_m)$

In this section we can introduce some preliminaries in $SU_{2,m}/S(U_2{\cdot}U_m)$ corresponding to the formulas given in [12] from the affection of the negative curvature tensor (3.8). Now let us contract Y and Z in the equation of Gauss (3.8) in section 3. Then the negativity of the curvature tensor for a real hypersurface M in $SU_{2,m}/S(U_2{\cdot}U_m)$ gives a Ricci tensor defined by

$$
\begin{aligned}
SX&=\sum\nolimits_{i=1}^{4m-1}R(X,e_i)e_i\\
&=-\frac{1}{2}\Big[(4m+10)X-3\eta(X)\xi-3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu\\
&\quad+\sum_{\nu=1}^{3}\{(\mathrm{Tr}\phi_\nu\phi)\phi_\nu\phi X-(\phi_\nu\phi)^2X\}\\
&\quad-\sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi X-\eta(X)\phi_\nu\phi\xi_\nu\}\\
&\quad-\sum_{\nu=1}^{3}\{(\mathrm{Tr}\ \phi_\nu\phi)\eta(X)-\eta(\phi_\nu\phi X)\}\xi_\nu\Big]+hAX-A^2X, \qquad (4.1)
\end{aligned}
$$

where h denotes the trace of the shape operator A of M in $SU_{2,m}/S(U_2{\cdot}U_m)$. From the formula $JJ_\nu=J_\nu J$, Tr $JJ_\nu=0$, $\nu=1,2,3$ we calculate the following for any basis $\{e_1,\cdots,e_{4m-1},N\}$ of the tangent space of $SU_{2,m}/S(U_2{\cdot}U_m)$

$$
\begin{aligned}
0&=\mathrm{Tr}\ JJ_\nu\\
&=\sum\nolimits_{k=1}^{4m-1}g(JJ_\nu e_k,e_k)+g(JJ_\nu N,N)\\
&=\mathrm{Tr}\ \phi\phi_\nu-\eta_\nu(\xi)-g(J_\nu N,JN)\\
&=\mathrm{Tr}\ \phi\phi_\nu-2\eta_\nu(\xi) \qquad (4.2)
\end{aligned}
$$

and

$$
\begin{aligned}
(\phi_\nu\phi)^2X&=\phi_\nu\phi(\phi\phi_\nu X-\eta_\nu(X)\xi+\eta(X)\xi_\nu)\\
&=\phi_\nu(-\phi_\nu X+\eta(\phi_\nu X)\xi)+\eta(X){\phi_\nu}^2\xi\\
&=X-\eta_\nu(X)\xi_\nu+\eta(\phi_\nu X)\phi_\nu\xi
\end{aligned}
$$

$$+\eta(X)\{-\xi+\eta_\nu(\xi)\xi_\nu\}. \tag{4.3}$$

Substituting (4.2) and (4.3) into (4.1), we have

$$\begin{aligned} SX = -\frac{1}{2}\Big[&(4m+7)X - 3\eta(X)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu \\ &+ \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi X - \eta(\phi_\nu X)\phi_\nu\xi - \eta(X)\eta_\nu(\xi)\xi_\nu\}\Big] \\ &+ hAX - A^2X. \end{aligned} \tag{4.4}$$

Remark 4.1. If a real hypersurface M in $SU_{2,m}/S(U_2{\cdot}U_m)$ is locally congruent to an open part of a tube around some totally geodesic $SU_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ or a horosphere whose center at infinity with $JX\in\mathfrak{J}X$ is singular, then the Reeb vector field $\xi=\xi_1$, and the mean curvature h should be constant. Also it can be easily checked that the above kind of tube or a horosphere satisfy $\mathcal{L}_\xi A=0$ (see [21]). Moreover, for the fact that $\xi=\xi_1$ they satisfy $\mathcal{L}_\xi\phi=0$, $\mathcal{L}_\xi\phi_1=0$, and

$$(\mathcal{L}_\xi\eta_2)\otimes\xi_2+\eta_2\otimes\mathcal{L}_\xi\xi_2+(\mathcal{L}_\xi\eta_3)\otimes\xi_3+\eta_3\otimes\mathcal{L}_\xi\xi_3=0$$

from their geometric properties in (3.1), (3.2), (3.3) and (3.4). In particular, the last formula mentioned above can be derived from $\mathcal{L}_\xi\xi_2=q_1(\xi)\xi_3-(\alpha-\beta)\xi_3$ and $\mathcal{L}_\xi\xi_3=-q_1(\xi)\xi_2+(\alpha-\beta)\xi_2$. Then by virtue of these properties, they naturally satisfy $\mathcal{L}_\xi S=0$, that is, the Ricci tensor is Reeb invariant.

In this section, we consider the converse problem. If the Ricci tensor of M in $SU_{2,m}/S(U_2{\cdot}U_m)$ is *Reeb invariant*, what can we say about such a hypersurface M. So in order to give a complete classification for M in $SU_{2,m}/S(U_2{\cdot}U_m)$ satisfying $\mathcal{L}_\xi S=0$, we want to compute the following

$$\begin{aligned} S\phi X = -\frac{1}{2}\Big[&(4m+7)\phi X - 3\sum_{\nu=1}^{3}\eta_\nu(\phi X)\xi_\nu \\ &+ \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi^2 X - \eta(\phi_\nu\phi X)\phi_\nu\xi - \eta(\phi X)\eta_\nu(\xi)\xi_\nu\}\Big] \\ &+ hA\phi X - A^2\phi X \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} \phi SX = -\frac{1}{2}\Big[&(4m+7)\phi X - 3\sum_{\nu=1}^{3}\eta_\nu(X)\phi\xi_\nu \\ &+ \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi\phi_\nu\phi X - \eta(\phi_\nu X)\phi\phi_\nu\xi - \eta(X)\eta_\nu(\xi)\phi\xi_\nu\}\Big] \\ &+ h\phi AX - \phi A^2X. \end{aligned} \tag{4.6}$$

Then from (4.5) and (4.6) it follows that

$$\begin{aligned} (\phi S - S\phi)X = {}& 2\sum_{\nu=1}^{3}\eta_\nu(X)\phi\xi_\nu - 2\sum_{\nu=1}^{3}\eta_\nu(\phi X)\xi_\nu + h(\phi A - A\phi)X \\ &- (\phi A^2 - A^2\phi)X. \end{aligned} \tag{4.7}$$

So we are able to calculate the following

$$\begin{aligned}\mathrm{Tr}\ (\phi S - S\phi)^2 = h\mathrm{Tr}\ (\phi A - A\phi)(\phi S - S\phi) - \mathrm{Tr}\ (\phi A^2 - A^2\phi)(\phi S - S\phi)\\ + 2\sum_{\nu=1}^{3}\mathrm{Tr}\ (\eta_\nu \otimes \phi\xi_\nu)(\phi S - S\phi)\\ - 2\sum_{\nu=1}^{3}\mathrm{Tr}\ (\eta\circ\phi\otimes\xi_\nu)(\phi S - S\phi).\end{aligned} \tag{4.8}$$

On the other hand, the terms in the right side of (4.8) respectively given by

$$\begin{aligned}\mathrm{Tr}\ (\eta_\nu\otimes\phi\xi_\nu)(\phi S - S\phi) &= \sum_i g(\eta_\nu((\phi S - S\phi)e_i)\phi\xi_\nu, e_i)\\ &= \sum_i g((\phi S - S\phi)e_i, \xi_\nu)g(\phi\xi_\nu, e_i) = g((\phi S - S\phi)\phi\xi_\nu, \xi_\nu)\\ &= -g(\phi\xi_\nu, (\phi S - S\phi)\xi_\nu)\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}\mathrm{Tr}\ (\eta_\nu\circ\phi\otimes\xi_\nu)(\phi S - S\phi) &= \sum_i g(\eta_\nu((\phi^2 S - \phi S\phi)e_i)\xi_\nu, e_i)\\ &= \eta_\nu((\phi^2 S - \phi S\phi)\xi_\nu) = -g((\phi S - S\phi)\xi_\nu, \phi\xi_\nu).\end{aligned} \tag{4.10}$$

Then by (4.9) and (4.10), the formula (4.8) becomes

$$\begin{aligned}\mathrm{Tr}\ (\phi S - S\phi)^2 &= h\mathrm{Tr}\ (\phi A - A\phi)(\phi S - S\phi) - \mathrm{Tr}\ (\phi A^2 - A^2\phi)(\phi S - S\phi)\\ &= -\mathrm{Tr}\ (\phi A^2 - A^2\phi)(\phi S - S\phi),\end{aligned}$$

where we have used $(\phi A - A\phi)S = S(\phi A - A\phi)$ from the symmetry of $\mathcal{L}_\xi S = 0$ and

$$\begin{aligned}\mathrm{Tr}\ (\phi A - A\phi)(\phi S - S\phi) &= \mathrm{Tr} S(\phi A - A\phi)\phi - \mathrm{Tr}(\phi A - A\phi)S\phi\\ &= \mathrm{Tr}(\phi A - A\phi)S\phi - \mathrm{Tr}(\phi A - A\phi)S\phi\\ &= 0\end{aligned}$$

From this, the right side becomes

$$\begin{aligned}\mathrm{Tr}\ (\phi A^2 - A^2\phi)(\phi S - S\phi) &= \mathrm{Tr}\phi A^2\phi S - \mathrm{Tr} A^2\phi^2 S - \mathrm{Tr}\phi A^2 S\phi\\ &\quad + \mathrm{Tr} A^2\phi S\phi\\ &= 2\mathrm{Tr}\ \phi A^2\phi S - \mathrm{Tr} A^2\phi^2 S - \mathrm{Tr}\phi A^2 S\phi.\end{aligned} \tag{4.11}$$

On the other hand, the symmetry of $\nabla_\xi S = \phi AS - S\phi A$, which is equivalent to $\mathcal{L}_\xi S = 0$, gives

$$(\phi A - A\phi)S = S(\phi A - A\phi),$$

where we have used for any X, Y in M

$$g((\phi AS - S\phi A)X, Y) = g((\phi AS - S\phi A)Y, X) = g((A\phi S - SA\phi)X, Y).$$

This implies

$$\phi A(\phi AS - S\phi A + SA\phi - A\phi S) = 0,$$

so that we know

$$\text{Tr } \phi ASA\phi = \text{Tr } \phi A^2\phi S, \tag{4.12}$$

because

$$\begin{aligned}\text{Tr}\phi A(\phi AS - S\phi A) &= \text{Tr}(\phi A)^2 S - \text{Tr}(\phi A)S(\phi A)\\ &= \text{Tr}(\phi A)^2 S - \text{Tr}(\phi A)^2 S.\end{aligned}$$

Then from (4.11) and (4.12) it follows that

$$\begin{aligned}\text{Tr } (\phi S - S\phi)^2 &= -\text{Tr } (\phi A^2 - A^2\phi)(\phi S - S\phi)\\ &= \text{Tr } \phi^2 SA^2 + \text{Tr } \phi^2 A^2 S - 2\text{Tr } \phi^2 ASA.\end{aligned} \tag{4.13}$$

On the other hand, the right side of (4.13) can be calculated term by term as follows:

$$\begin{aligned}\text{Tr } \phi^2 ASA &= \text{Tr } (-ASA + \eta(ASA)\xi) = -\text{Tr } ASA + \eta(ASA\xi),\\ \text{Tr } \phi^2 SA^2 &= \text{Tr } (-SA^2 + \eta(SA^2)\xi) = -\text{Tr } SA^2 + \eta(SA^2\xi),\end{aligned}$$

and

$$\text{Tr } \phi^2 A^2 A = \text{Tr } (-A^2 S + \eta(A^2 S)\xi) = -\text{Tr } A^2 S + \eta(A^2 S\xi).$$

Substituting these formulas into (4.13) gives the following

$$\begin{aligned}\text{Tr } (\phi S - S\phi)^2 &= -\text{Tr } SA^2 + \eta(SA^2\xi) - \text{Tr } A^2 S + \eta(A^2 S\xi)\\ &\quad + 2\text{Tr } ASA - 2\eta(ASA\xi)\\ &= 2\eta(SA^2\xi) - 2\eta(ASA\xi).\end{aligned} \tag{4.14}$$

Now from the expression of the Ricci tensor (4.4) for the Reeb vector field ξ we have the following respectively

$$S\xi = -2(m+1)\xi + 2\sum_{\nu=1}^{3}\eta_\nu(\xi)\xi_\nu + hA\xi - A^2\xi,$$

and

$$\begin{aligned}\eta(SA^2\xi) &= -2(m+1)\|A\xi\|^2 + 2\sum_{\nu=1}^{3}\eta_\nu(\xi)g(\xi_\nu, A^2\xi)\\ &\quad + hg(A\xi,\xi) - g(A^2\xi, A^2\xi),\\ \eta(ASA\xi) &= g(SA\xi, A\xi)\\ &= -\frac{1}{2}\Big[(4m+7)g(A\xi, A\xi) - 3\eta(A\xi)^2 - 3\sum_{\nu=1}^{3}\eta_\nu(A\xi)^2\end{aligned}$$

$$+\sum_{\nu=1}^{3}\{\eta_\nu(\xi)g(\phi_\nu\phi A\xi, A\xi) - \eta(\phi_\nu A\xi)g(\phi_\nu\xi, A\xi)$$
$$- \eta(A\xi)\eta_\nu(\xi)\eta_\nu(A\xi)\}\Big] + hg(A^2\xi, A\xi) - g(A^3\xi, A\xi).$$

Then the formula (4.14) for M in $SU_{2,m}/S(U_2{\cdot}U_m)$ becomes

$$\begin{aligned}\mathrm{Tr}\ (\phi S - S\phi)^2 &= 2\eta(SA^2\xi) - 2\eta(ASA\xi)\\ &= 3\|A\xi\|^2 - 3\eta(A\xi)^2 - 3\sum_{\nu=1}^{3}\eta_\nu(A\xi)^2\\ &\quad + 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(A^2\xi) + \sum_{\nu=1}^{3}\Big\{\eta_\nu(\xi)g(\phi_\nu\phi A\xi, A\xi)\\ &\quad + \eta(\phi_\nu A\xi)^2 - \eta(A\xi)\eta_\nu(\xi)\eta_\nu(A\xi)\Big\}.\end{aligned} \tag{4.15}$$

From this, together with (3.2), (3.3), (3.4) and the notion of Hopf, the right side of (4.15) should be vanishing as follows:

$$\mathrm{Tr}\ (\phi S - S\phi)^2 = -3\alpha^2\sum_{\nu=1}^{3}\eta_\nu(\xi)^2 + 4\alpha^2\sum_{\nu=1}^{3}\eta_\nu(\xi)^2 - \alpha^2\sum_{\nu=1}^{3}\eta_\nu(\xi)^2 = 0$$

if we assume that a Hopf hypersurface M in $SU_{2,m}/S(U_2{\cdot}U_m)$ satisfies $\mathcal{L}_\xi S = 0$. This gives that the Ricci tensor S commutes with the structure tensor ϕ, that is, $S\phi = \phi S$. Then by Theorem B we can assert our main result. This gives a complete proof of our Main Theorem.

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